Strong Convergence on Weakly Logarithmic Combinatorial Assemblies

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Abstract

We deal with the random combinatorial structures called assemblies. By weakening the logarithmic condition which assures regularity of the number of components of a given order, we extend the notion of logarithmic assemblies. Using the author's analytic approach, we generalize the so-called Fundamental Lemma giving independent process approximation in the total variation distance of the component structure of an assembly. To evaluate the influence of strongly dependent large components, we obtain estimates of the appropriate conditional probabilities by unconditioned ones. These estimates are applied to examine additive functions defined on such a class of structures. Some analogs of Major's and Feller's theorems which concern almost sure behavior of sums of independent random variables are proved.

1 Introduction

In part, this work was stimulated by a critical remark made by R. Arratia, A.D. Barbour and S. Tavaré [1] about analytic methods applied in the theory of random combinatorial structures. On page 1622 they wrote: *In contrast* (to their method), *the complex analytic approaches typically require conditions to be satisfied that can be verified in the well-known examples, but which are difficult to express directly in terms of the basic parameters of the structures.* Such was the criticism to the method cultivated in the papers by P. Flajolet and M. Soria [8] and J. Hansen [11]. The works written by D. Stark [26] and [27] could be added to this list as well. Indeed, the conditions posed on the generating series of structure classes have some disadvantages.

The authors of [1] did not notice the broader possibilities hidden in the analytic approach proposed in our papers [13], [14], [17], and refined in [6] and [20]. So far, this approach was applied to obtain asymptotic formulas for some Fourier transforms of distributions. That led to general one-dimensional

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limit theorems, including the optimal remainder term estimates. In this regard, apart from the above mentioned, the papers by V. Zacharovas [28], [29], and [30] were noticeable. On the other hand, there exist a lot of works dealing with the deeper total variation approximation (see, for instance, [2] and the references therein). The main goal of the present paper is to demonstrate that such total variation approximations can be obtained by our method and, at the same time, under more general conditions possed on *the basic parameters* of the structures. For simplicity, we confine ourselves to classes of *assemblies* or *abelian partitional complexes* (see [9]). For completeness, we recall the definition and some properties which can be found in [2].

Let σ be a set of $n \geq 1$ points, partitioned into subsets so that there are $k_j(\sigma) > 0$ subsets of size j, $1 \leq j \leq n$ and $\bar{k}(\sigma) := \big(k_1(\sigma), \dots, k_n(\sigma)\big)$. If $\ell(\bar{s}) := 1s_1 + \cdots + ns_n$, where $\bar{s} = (s_1, \dots, s_n) \in \mathbb{Z}_+^n$, then $\ell\big(\bar{k}(\sigma)\big) = n$. Assume that in each such subset of size $1 \leq j \leq n$ by some rule one of $0 < m_j < \infty$ possible structures can be chosen. A subset with a structure is a *component* of σ, and the set σ itself is called an assembly [2]. Using all possible partitions of σ and the same rule to define a structure in a component, we get the class $\mathcal{A}_n$ of assemblies of size n. Let $\mathcal{A}_0$ be comprised of the empty set. The union

$$\mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n \cup \cdots$$

forms the whole class of assemblies. Its basic parameters appear in the conditions posed on the sequence m_j, $j \geq 1$.

There are

$$n! \prod_{j=1}^{n} \left(\frac{1}{j!}\right)^{s_j} \frac{1}{s_j!}$$

ways to partition an n-set into subsets, so that $\bar{k}(\sigma) = \bar{s}$ if $\ell(\bar{s}) = n$ and $\bar{s} \in \mathbb{Z}_+^n$. Hence, there are

$$Q_n(\bar{s}) := n! \prod_{j=1}^{n} \left(\frac{m_j}{j!}\right)^{s_j} \frac{1}{s_j!}$$

assemblies with the component vector $\bar{k}(\sigma) = \bar{s}$, and the total number of them in the class $\mathcal{A}_n$ equals

$$|\mathcal{A}_n| = \sum_{\ell(\bar{s})=n} Q_n(\bar{s}).$$

On the class $\mathcal{A}_n$, one can define the uniform probability measure denoted by

$$\nu_n(\dots) = |\mathcal{A}_n|^{-1} |\{\sigma \in \mathcal{A}_n, \dots\}|.$$

From now $\sigma \in \mathcal{A}_n$ is an elementary event. Following the tradition of probabilistic number theory and in contrast to [2], we prefer to leave it defining random variables (r.vs) on $\mathcal{A}_n$. The component vector $\bar{k}(\sigma)$ has the following distribution:

$$\nu_n(\bar{k}(\sigma) = \bar{s}) = \mathbf{1}\{\ell(\bar{s}) = n\} \frac{n!}{|\mathcal{A}_n|} \prod_{j=1}^{n} \frac{1}{s_j!} \left(\frac{m_j}{j!}\right)^{s_j},$$

where $\bar{s} = (s_1, \ldots, s_n) \in \mathbb{Z}_+^n$. This leads to the *Conditioning Relation* (see [2], page 48)

$$\nu_n(\bar{k}(\sigma) = \bar{s}) = P\big(\bar{\xi} = \bar{s} | \ell(\bar{\xi}) = n\big), \tag{1}$$

where $\bar{\xi} := (\xi_1, \ldots, \xi_n)$ and $\xi_j, j \geq 1$, are mutually independent Poisson r.vs defined on some probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with $\mathbf{E}\xi_j = u^j m_j/j!$, $j \geq 1$, where $u > 0$ is an arbitrary number.

The so-called *Logarithmic Condition* (see [2]) in the case of assemblies requires that

$$m_j/j! \sim \theta y^j/j$$

for some constants $y > 0$ and $\theta > 0$ as $j \to \infty$. Under this condition, it is natural and technically convenient to take $u = y^{-1}$, which yields the relation $\mathbf{E}\xi_j \sim \theta/j$ as $j \to \infty$.

Generalizing the Ewens probability in the symmetric group of permutations, the author in [17] and [20] examined random assemblies taken with weighted frequencies. The research was extended by V. Zacharovas [31]. Going along this path, one can take a positive sequence w_j, $j \geq 1$, and define

$$w(\sigma) = \prod_{j=1}^{n} w_j^{k_j(\sigma)}, \qquad W_n = \sum_{\sigma \in \mathcal{A}_n} w(\sigma).$$

Further, one can introduce the probability measure $\nu_n^{(w)}$ on $\mathcal{A}_n$ by

$$\nu_n^{(w)}\big(\{\sigma\}\big) = w(\sigma)/W_n, \quad \sigma \in \mathcal{A}_n.$$

Conditioning Relation (1) still holds for $\nu_n^{(w)}$ instead of ν_n with the poissonian random vector $\bar{\xi}$ provided that $\mathbf{E}\xi_j = u^j m_j w_j/j!$, where $j \geq 1$ and $u > 0$ is an arbitrary constant. Having all this in mind, we extend the logarithmic class of assemblies discussed in [2] and in many previous papers.

Definition. *Let $n \geq 1$ and let μ_n be a probability measure on $\mathcal{A}_n$. The pair $\big(\mathcal{A}_n, \mu_n\big)$ will be called weakly logarithmic if there exists a random vector $\bar{\xi} = (\xi_1, \ldots, \xi_n)$ with mutually independent poissonian coordinates such that*

$$\mu_n(\bar{k}(\sigma) = \bar{s}) = P\big(\bar{\xi} = \bar{s} | \ell(\bar{\xi}) = n\big)$$

for each $\bar{s} \in \mathbb{Z}_+^n$ and

$$\frac{\theta'}{j} \leq \lambda_j := \mathbf{E}\xi_j \leq \frac{\theta''}{j} \tag{2}$$

uniformly in $j \geq 1$ for some positive constants θ' and θ''.

In our notation, the *logarithmic assemblies* are characterized by the condition $\lambda_j \sim \theta/j$ as $j \to \infty$, where $\theta > 0$ is a constant (see [2]).

The main result of this paper is the following total variation approximation. Let $\mathcal{L}(X)$ be the distribution of a r.v. X. Afterwards the index r, $1 \leq r \leq n$, added to the vectors $\bar{k}(\sigma)$ and $\bar{\xi}$ will denote that only the first r coordinates are taken. Let $x_+ = \max\{x, 0\}$ for $x \in \mathbb{R}$ and $\ll$ be an analog of the symbol $\mathrm{O}(\cdot)$.

Theorem (Fundamental Lemma). *Let $(\mathcal{A}_n, \mu_n)$ be weakly logarithmic. There exist positive constants c_1 and c_2 depending on θ' and such that*

$$\rho_{TV}\Big(\mathcal{L}\big(\bar{k}_r(\sigma)\big), \mathcal{L}(\bar{\xi}_r)\Big) := \sum_{\bar{s}\in\mathbb{Z}_+^r} \Big(\mu_n\big(\bar{k}_r(\sigma)=\bar{s}\big) - P(\bar{\xi}_r=\bar{s})\Big)_+ \ll \Big(\frac{r}{n}\Big)^{c_1} \tag{3}$$

uniformly in $1 \le r \le c_2 n$. The constant in $\ll$ depends on θ' and θ'' only.

Adopting I. Z. Ruzsa's idea going back to probabilistic number theory (see [25]), we [15] observed that some conditional discrete probabilities can be estimated by appropriate unconditional ones. This led to upper estimates of the distributions $\mathcal{L}\big(\bar{k}(\sigma)\big)$ of the cycle structure vector $\bar{k}(\sigma)$ of a random permutation σ under the uniform probability defined on the symmetric group. In the joint paper with G.J. Babu [3], the idea was extended to permutations taken with the Ewens probability and later, jointly with J. Norkūnienė [21], we adopted it for logarithmic assemblies. We now develop the same principle for weakly logarithmic assemblies.

Firstly, we introduce some notation in the semi-lattice $\mathbb{Z}_+^n$ taken from the theory of euclidean spaces. For two vectors $\bar{s} = (s_1, \dots, s_n)$ and $\bar{t} = (t_1, \dots, t_n)$, we set $\bar{s} \perp \bar{t}$ if $s_1t_1 + \cdots + s_nt_n = 0$ and write $\bar{s} \le \bar{t}$ if $s_j \le t_j$ for each $j \le n$. Further, we adopt the notation $\bar{s} \parallel \bar{t}$ for the expression "$\bar{s}$ exactly enters $\bar{t}$" which means that $\bar{s} \le \bar{t}$ and $\bar{s} \perp \bar{t} - \bar{s}$. For arbitrary subset $U \subset \mathbb{Z}_+^n$, we define its extension

$$V = V(U) = \big\{\bar{s} = \bar{t^1} + \bar{t^2} - \bar{t^3} :\ \bar{t^1}, \bar{t^2}, \bar{t^3} \in U,\ \bar{t^1} \perp (\bar{t^2} - \bar{t^3}),\ \bar{t^3} \parallel \bar{t^2}\big\}. \tag{4}$$

Set also $\overline{A} = \mathbb{Z}_+^n \setminus A$ and $\theta = \min\{1, \theta'\}$.

Theorem 1. *Let $(\mathcal{A}_n, \mu_n)$ be weakly logarithmic and $\bar{\xi}$ be the poissonian random vector introduced in Definition. For arbitrary $U \in \mathbb{Z}_+^n$,*

$$\mu_n\big(\bar{k}(\sigma) \in \overline{V}\big) = P\big(\bar{\xi} \in \overline{V} \,|\, \ell(\bar{\xi}) = n\big) \ll P^{\theta}(\bar{\xi} \in \overline{U}) + \mathbf{1}\{\theta < 1\} n^{-\theta},$$

where the implicit constants depend on θ' and θ'' only.

The claim of Theorem 1 becomes more transparent when applied to the value distributions of additive functions. We demonstrate this in a fairly general context. Let $(\mathbb{G}, +)$ be an abelian group and $h_j(s)$, $j \in \mathbb{N}$, $s \in \mathbb{Z}_+$, be a two-dimensional sequence in $\mathbb{G}$ satisfying the condition $h_j(0) = 0$ for each $j \ge 1$. Then we can define an *additive function* $h\colon \mathcal{A}_n \to \mathbb{G}$ by

$$h(\sigma) = \sum_{j \le n} h_j\big(k_j(\sigma)\big). \tag{5}$$

If $h_j(s) = a_j s$ for some $a_j \in \mathbb{G}$, where $j \in \mathbb{N}$ and $s \in \mathbb{Z}_+$, then the function h is called *completely additive.*

Corollary 1. *Let* $(\mathbb{G},+)$ *be an abelian group and* $h\colon \mathcal{A}_n \to \mathbb{G}$ *be an additive function. Uniformly in* $A \subset \mathbb{G}$,

$$\mu_n\big(h(\sigma) \notin A + A - A\big) \ll P^\theta\Bigg(\sum_{j\le n} h_j(\xi_j) \notin A\Bigg) + \mathbf{1}\{\theta < 1\}n^{-\theta}.$$

Corollary 2. *Let* $h\colon \mathcal{A}_n \to \mathbb{R}$ *be an additive function. Uniformly in* $a \in \mathbb{R}$ *and* $u \ge 0$,

$$\mu_n\big(|h(\sigma) - a| \ge u\big) \ll P^\theta\Bigg(\Bigg|\sum_{j\le n} h_j(\xi_j) - a\Bigg| \ge u/3\Bigg) + \mathbf{1}\{\theta < 1\}n^{-\theta}.$$

As in the case of logarithmic assemblies, Fundamental Lemma and Theorem 1 can be used to prove general limit theorems for additive functions defined on $\mathcal{A}_n$. One can deal with the one-dimensional case (see, for instance, [2], Section 8.5) or examine the weak convergence of random combinatorial processes (see [3], [4], [5], [16], and [2], Section 8.1). This approach can be applied to examine the strong convergence. Extending papers [18] and [23], we now obtain an analog of the functional law of iterated logarithm. It can be compared with Major's [12] result for i.r.vs, generalizing the celebrated Strassen's theorem.

It is worth stressing that we deal with random variables which are defined on a sequence of probability spaces, not on a fixed space. This raises the first obstacle to be overcome; therefore, we adopt some basic definitions.

Let (S,d) be a separable metric space. Assume that $X, X_1, X_2, \dots, X_n$ are S-valued random variables all defined on the probability space $\{\Omega_n, \mathcal{F}_n, P_n\}$. Denote by $d(Y,A) := \inf\{d(Y,Z) : Z \in A\}, \quad A \subset S,\ Y \in S$, the distance from Y to A. We say that X_m converges to X $\{P_n\}$-almost surely ($\{P_n\}$-a.s.), if for each $\varepsilon > 0$

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} P_n\big(\max_{n_1\le m\le n} d(X_m, X) \ge \varepsilon\big) = 0.$$

If $P_n = P$ does not depend on n, our definition agrees with that of classical almost sure convergence (see [24], Chapter X). A compact set $A \subset S$ is called a *cluster* for the sequence X_m if, for each $\varepsilon > 0$ and each $Y \in A$,

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} P_n\big(\max_{n_1\le m\le n} d(X_m, A) \ge \varepsilon\big) = 0$$

and

$$\lim_{n_1\to\infty} \liminf_{n\to\infty} P_n\big(\min_{n_1\le m\le n} d(X_m, Y) < \varepsilon\big) = 1.$$

We denote the last two relations, by

$$X_m \Rightarrow A, \qquad (\{P_n\}\text{-a.s.})$$

Let $C[0,1]$ be the Banach space of continuous functions on the interval $[0,1]$ with the supremum distance $\rho(\cdot,\cdot)$. The set of absolutely continuous functions g such that $g(0) = 0$ and

$$\int_0^1 (g'(t))^2 dt \le 1$$

is called the *Strassen set* $\mathcal{K}$. We shall show that it is the cluster set of some combinatorial processes constructed using partial sums

$$h(\sigma, m) := \sum_{j \le m} h_j\big(k_j(\sigma)\big),$$

where $h_j(s) \in \mathbb{R}$ and $1 \le m \le n$. Set $a_j = h_j(1)$,

$$A(m) := \sum_{j=1}^{m} a_j(1 - \mathrm{e}^{-\lambda_j}), \qquad B^2(m) := \sum_{j=1}^{m} a_j^2 e^{-\lambda_j}\big(1 - e^{-\lambda_j}\big),$$

and $\beta(m) = B(m)\sqrt{2LLB(m)}$, where $1 \le m \le n$. We denote by $u_m(\sigma, t)$ the polygonal line joining the points

$$(0,0), \qquad \big(B^2(i), h(\sigma, i) - A(i)\big), \quad 1 \le i \le m,$$

and set

$$U_m(\sigma, t) = \beta(m)^{-1} u_m(\sigma, B^2(m)t), \qquad \sigma \in \mathcal{A}_n,\ 0 \le t \le 1,$$

for $1 \le m \le n$. The following result generalizes the cases examined in [15], [22], and [23].

Theorem 2. *Let $(\mathcal{A}_n, \mu_n)$ be weakly logarithmic. If $B(n) \to \infty$ and*

$$a_j = \mathrm{o}\bigg(\frac{B(j)}{\sqrt{LLB(j)}}\bigg), \qquad j \to \infty, \tag{6}$$

then

$$U_m(\sigma, \cdot) \Rightarrow \mathcal{K} \qquad (\{\mu_n\}\text{-a.s.}). \tag{7}$$

Applying continuous functionals defined on the space $C[0,1]$, we derive partial cases of the last theorem.

Corollary 3. *Let the conditions of Theorem 3 be satisfied. The following relations hold $\{\mu_n\}$-a.s.*

(i) $U_m(1) \Rightarrow [-1, 1]$;

(ii) $\big(U_m(\sigma, 1/2), U_m(\sigma, 1)\big) \Rightarrow \{(u, v) \in \mathbb{R}^2 :\ u^2 + (v-u)^2 \le 1/2\}$;

(iii) *if* $U_{m'}(\sigma, 1/2) \Rightarrow \sqrt{2}/2$ *for some subsequence* $m' \to \infty$, *then* $U_{m'}(\sigma, \cdot) \Rightarrow g_1$, *where*

$$g_1(t) = \begin{cases} t\sqrt{2} & \text{if} \quad 0 \le t \le 1/2, \\ \sqrt{2}/2 & \text{if} \quad 1/2 \le t \le 1; \end{cases}$$

(iv) if $U_{m'}(\sigma,1/2) \Rightarrow 1/2$ and $U_{m'}(\sigma,1) \Rightarrow 0$ for some subsequence $m' \to \infty$, then $U_{m'}(\sigma,\cdot) \Rightarrow g_2$, where

$$g_2(t) = \begin{cases} t & \text{if} \quad 0 \le t \le 1/2, \\ 1-t & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

Using other more sophisticated functionals (see, e.g., [10], Chapter I), one can proceed in a similar manner. Claim (i) includes the assertion that

$$\big|h(\sigma,m) - A(m)\big| \le (1+\varepsilon)\beta_m$$

holds uniformly in m, $n_1 \le m \le n$, for asymptotically almost all $\sigma \in \mathcal{A}_n$ as n and n_1 tend to infinity. Moreover, it shows that the upper bound is sharp apart from the term $\varepsilon\beta(m)$. An idea how to improve this error goes back to W. Feller's paper [7]. It has been exploited by the author [18] in the case of a special additive function defined on permutations. Recently, that paper was generalized for the logarithmic assemblies [21]. We now formulate a more general result.

We say that an increasing sequence ψ_m, $m \ge 1$, belongs to the *upper class* Ψ^+ (respectively, *the lower class* Ψ^-) if

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} \mu_n\Big(\max_{n_1 \le m \le n} \psi_m^{-1} \big|h(\sigma,m) - A(m)\big| \ge 1 \Big) = 0, \tag{8}$$

$$\bigg(\lim_{n_1\to\infty} \liminf_{n\to\infty} \mu_n\Big(\max_{n_1 \le m \le n} \psi_m^{-1} \big|h(\sigma,m) - A(m)\big| \ge 1 \Big) = 1 \bigg).$$

Theorem 3. *Let $(\mathcal{A}_n, \mu_n)$ be weakly logarithmic and $B(n) \to \infty$. Assume that a positive sequence $\phi_n \to \infty$ is such that*

$$a_j = \mathrm{O}\bigg(\frac{B(j)}{\phi_j^3}\bigg), \quad j \ge 1. \tag{9}$$

If the series

$$\sum_{j=1}^{\infty} \frac{a_j^2 \phi_j}{jB^2(j)} \, \mathrm{e}^{-\phi_j^2/2} \tag{10}$$

converges, then $B(m)\phi_m \in \Psi^+$. If series (10) diverges, then $B(m)\phi_m \in \Psi^-$.

Since the series

$$\sum_{j=1}^{\infty} \frac{a_j^2}{j} \frac{(LLB(j))^{1/2}}{B^2(j)(LB(j))^{1+x}}$$

converges for $x = \varepsilon$ and diverges for $x = -\varepsilon$, the last theorem implies (i) in Corollary 3 under a bit stronger condition. To illustrate Theorem 3, let $\gamma_{2m}^2(\pm\varepsilon) := 2(1\pm\varepsilon)L_2B(m)$,

$$\gamma_{3m}^2(\pm\varepsilon)/2 := L_2B(m) + \frac{3}{2}(1\pm\varepsilon)L_3B(m),$$

and

$$\gamma^2_{sm}(\pm\varepsilon)/2 := L_2B(m) + \frac{3}{2}L_3B(m) + L_4B(m) + \cdots + (1\pm\varepsilon)L_sB(m)$$

for $s \geq 4$.

Corollary 4. *Under the conditions of Theorem 3, we have*

$$B(m)\gamma_{sm}(\varepsilon) \in \Psi^+$$

and

$$B(m)\gamma_{sm}(-\varepsilon) \in \Psi^-$$

for each $s \geq 2$.

More corollaries, as in the case of the logarithmic assemblies (see [21]), could be further formulated. The main argument in deriving Theorems 2 and 3 is the same; therefore, we will omit the proofs of the second result and its corollaries. The technical details in the case of logarithmic assemblies can be found in [21]. Finally, we observe that by substituting r.vs ξ_j, $1 \leq j \leq n$, by appropriate independent geometrically distributed and negative binomial r.vs, one can similarly extend the logarithmic classes of additive arithmetical semigroups and weighted multisets (see [2]).

2 Proof of the Fundamental Lemma

The first lemma reduces the problem to a one-dimensional case. For $\bar{s} = (s_1, \ldots, s_n)$, set $\ell_{ij}(\bar{s}) = (i+1)s_{i+1} + \cdots + js_j$ if $0 \leq i < j \leq n$. Moreover, let $\ell_r(\bar{s}) := \ell_{0r}(\bar{s})$, where $1 \leq r \leq n$. Then $\ell_n(\bar{s}) = \ell(\bar{s})$.

Lemma 1. *We have*

$$\begin{aligned}\rho_{TV}\Big(\mathcal{L}\big(\bar{k}_r(\sigma)\big), \mathcal{L}(\bar{\xi}_r)\Big) &= \rho_{TV}\Big(\mathcal{L}\big(\bar{\xi}_r \big| \ell(\bar{\xi}) = n\big), \mathcal{L}(\bar{\xi}_r)\Big) \\ = \sum_{m \in \mathbb{Z}_+} P\big(\ell_r(\bar{\xi}) = m\big)&\bigg(1 - \frac{P\big(\ell_{rn}(\bar{\xi}) = n-m\big)}{P\big(\ell(\bar{\xi}) = n\big)}\bigg)_+ \end{aligned} \qquad (11)$$

Proof See [2], p. 60.

Consequently, the ratio of probabilities on the right-hand side in (11) is now the main objective. So far, the authors [1], assuming the Logarithmic Condition, kept obtaining the limit approximations as $n \to \infty$ for either of the probabilities, and then showing their equivalence in a fairly large region for m. The limiting behavior of the probabilities can be rather complicated for weakly logarithmic assemblies but, as we will show in the sequel, the ratio of probabilities in (11) is regular. Since

$$P\big(\ell_{rn}(\bar{\xi}) = m\big) = \frac{1}{2\pi i}\int_{|z|=1} \frac{1}{z^m} \exp\bigg\{\sum_{r<j\leq n} \lambda_j(z^j - 1)\bigg\} \mathrm{d}z, \qquad (12)$$

one can apply our analytic technique (see [17] or [20]) which has been elaborated to compare the Taylor coefficients of two power series. Namely, if $d_j \in \mathbb{R}_+$ and $f_j \in \mathbb{C}$, $1 \le j \le n$, are two sequences, the latter maybe depending on n or other parameters, and

$$D(z) := \exp\Big\{ \sum_{j\le n} \frac{d_j}{j} z^j \Big\} =: \sum_{s=0}^{\infty} D_s z^s,$$

$$F(z) := \exp\Big\{ \sum_{j\le n} \frac{f_j}{j} z^j \Big\} =: \sum_{s=0}^{\infty} F_s z^s,$$

then, under certain conditions, we have obtained asymptotic formulas for F_n/D_n as $n \to \infty$. As in [17], we now also assume the inequalities

$$d' \le d_j \le d'' \tag{13}$$

for all $1 \le j \le n$ and some positive constants $d' \le d''$. In our case, f_j are very special; therefore, we can simplify the previous argument and get rid of (2.4) in [17]. The goal now is to find the ratio F_m/D_n preserving some uniformity.

Set, for brevity,

$$e_r = \exp\Big\{ -\sum_{j\le r} \frac{d_j}{j} \Big\}.$$

Proposition 1. *Assume that the sequence d_j, $1 \le j \le n$, satisfies condition (13). For $0 \le r \le n$, set $f_j = d_j$ if $r < j \le n$ and $f_j = 0$ if $j \le r$. Let $0 \le \eta \le 1/2$ and $1/n \le \delta \le 1/2$ be arbitrary. There exists a positive constant c depending on d' only such that*

$$F_m/(e_r D_n) - 1 \ll \big(\eta + (r/n)\mathbf{1}\{r \ge 1\}\big)\delta^{-1} + \delta^c$$

uniformly in

$$0 \le r \le \delta n, \qquad n(1-\eta) \le m \le n. \tag{14}$$

Here and in the proof of this claim, the constant in $\ll$ depends on d' and d'' only.

We will use the following notation. Let K, $1 \le \delta n < K \le n$, be a parameter to be chosen later. For a fixed $0 < \alpha < 1$, we introduce the functions

$$q(z) := \sum_{r<j\le n} d_j z^{j-1}, \qquad G_1(z) = \exp\Big\{ \alpha \sum_{r<j\le K} \frac{d_j}{j} z^j \Big\},$$

$$G_2(z) = \exp\Big\{ -\alpha \sum_{K<j\le n} \frac{d_j}{j} z^j \Big\}, \qquad G_3(z) = F^{\alpha}(z) - G_1(z).$$

We denote by $[z^k]U(z)$ the kth Taylor coefficient of an analytic at zero function $U(z)$. Observe that

$$[z^k]G_3(z) \le [z^k]F^{\alpha}(z), \quad k \ge 0, \tag{15}$$

where $a_j = d_j$ if $r < j \le n$, and $a_j = 0$ otherwise. Set further $T = (\delta n)^{-1}$,

$$\Delta = \{z = e^{it} :\ T < |t| \le \pi\}, \qquad \Delta_0 = \{z = e^{it} :\ |t| \le T\}.$$

Seeking F_m, we start from the following identity

$$\begin{aligned} F_m &= \frac{1}{2\pi i m} \int_{|z|=1} \frac{F'(z)}{z^m}\,\mathrm{d}z \\ &= \frac{1}{2\pi i m} \Big(\int_{\Delta_0} + \int_{\Delta} \Big) \frac{F'(z)\big(1 - G_2(z)\big)}{z^m}\,\mathrm{d}z \\ &\quad + \frac{1}{2\pi i m} \int_{|z|=1} \frac{F'(z) G_2(z)}{z^m}\,\mathrm{d}z =: J_0 + J_1 + J_2. \end{aligned} \tag{16}$$

In what follows, we estimate the integrals J_1 and J_2 and, changing the integrand, reduce J_0 to the main term of an asymptotical formula for D_n. The proof of Proposition 1 consists of a few lemmas.

Lemma 2. *We have*

$$D(1)n^{-1} \ll D_n \ll D(1)n^{-1}$$

for all $n \ge 1$.

Proof. This is Lemma 3.1 from [17].

Lemma 3. *If* $0 < \alpha < 1$ *and* $\delta n \ge 1$, *then*

$$J_2 \ll D_n e_r (K/n)^{\alpha d'}$$

uniformly in $n/2 \le m \le n$.

Proof. For brevity, let

$$u_s := [z^s] G_1(z), \qquad v_l := [z^l] F^{1-\alpha}(z), \quad s, l \ge 0.$$

Since

$$F'(z) G_2(z) = q(z) G_1(z) F^{1-\alpha}(z),$$

from Cauchy's formula, we have

$$J_2 = \frac{1}{2\pi i m} \int_{|z|=1} q(z) G_1(z) F^{1-\alpha}(z) \frac{\mathrm{d}z}{z^m} = \frac{1}{m} \sum_{r<j\le m} d_j \sum_{s+l=m-j} u_s v_l.$$

Hence, by condition (13),

$$\begin{aligned} J_2 &\le \frac{2d''}{n} \sum_{s \le n} u_s \sum_{l \le n} v_l \\ &\le \frac{2d''}{n} F^{1-\alpha}(1) G_1(1) = \frac{2d'' F(1)}{n} \exp\Big\{ -\alpha \sum_{K<j\le n} \frac{d_j}{j} \Big\} \\ &\ll D_n e_r (K/n)^{\alpha d'}. \end{aligned}$$

In the last step we used Lemma 2.

The lemma is proved.

Lemma 4. *Let $\delta n \geq 1$. Then*

$$\max_{T \leq |t| \leq \pi} |F(\mathrm{e}^{it})| \ll e_r D(1) \delta^{d'}$$

uniformly in $0 \leq r \leq \delta n$.

Proof. By definition,

$$\begin{aligned} \frac{|F(\mathrm{e}^{it})|}{D(1)} &= e_r \frac{|F(\mathrm{e}^{it})|}{F(1)} = e_r \exp\Big\{ \sum_{r<j\leq n} \frac{d_j(\cos tj - 1)}{j} \Big\} \\ &\leq e_r \exp\Big\{ d' \sum_{\delta n < j \leq n} \frac{\cos tj - 1}{j} \Big\} \end{aligned} \tag{17}$$

uniformly in $0 \leq r \leq \delta n$. We now use the relation

$$S(x,t) := \sum_{j \leq x} \frac{\cos tj - 1}{j} = \log \min \Big\{ 1, \frac{2\pi}{x|t|} \Big\} + \mathrm{O}(1),$$

valid for all $x \geq 1$ and $|t| \leq \pi$. It shows that $S(\delta n, t) \ll 1$ for $T = (\delta n)^{-1} \leq |t| \leq \pi$. Hence, for such t,

$$S(n,t) - S(\delta n, t) \leq S(n,T) + \mathrm{O}(1) = \log \delta + \mathrm{O}(1).$$

This yields the desired claim.

Lemma 5. *Let $0 < \alpha < 1$ be arbitrary and $\delta n \geq 1$. Then*

$$J_1 \ll \frac{e_r n D_n}{K} \delta^{d'(1-\alpha)}$$

uniformly in $n/2 \leq m \leq n$ and $0 \leq r \leq \delta n$.

Proof. Recalling the previous notation, we can rewrite

$$J_1 = \frac{1}{2\pi i m} \int_{\Delta} q(z) F^{1-\alpha}(z) G_3(z) \frac{\mathrm{d}z}{z^m}.$$

Hence, by Lemma 4,

$$\begin{aligned} J_1 &\ll n^{-1} \max_{z \in \Delta} |F(z)|^{1-\alpha} \int_{|z|=1} \big|q(z)\big| \big|G_3(z)\big| |\mathrm{d}z| \\ &\ll n^{-1} \Big(e_r D(1) \delta^{d'} \Big)^{1-\alpha} \Big(\int_{|z|=1} \big|q(z)\big|^2 |\mathrm{d}z| \Big)^{1/2} \\ &\qquad \times \Big(\int_{|z|=1} |G_3(z)|^2 |\mathrm{d}z| \Big)^{1/2}. \end{aligned}$$

By Parseval's equality,

$$\int_{|z|=1} \big|q(z)\big|^2 |\mathrm{d}z| = 2\pi \sum_{r<j\leq n} d_j^2 \leq 2\pi (d'')^2 n$$

and, recalling (15),

$$
\begin{aligned}
&\int_{|z|=1} |G_3(z)|^2 |\mathrm{d}z| \le 2\pi \sum_{l>K} \big([z^l]G_3(z)\big)^2 \\
\le\;& \frac{2\pi}{K^2}\sum_{l=1}^{\infty} l^2\big([z^l]F^\alpha(z)\big)^2 \ll \frac{1}{K^2}\int_{|z|=1}\big|(F^\alpha(z))'\big|^2|\mathrm{d}z| \\
\ll\;& \frac{(e_rD(1))^{2\alpha}}{K^2}\int_{|z|=1}|q(z)|^2|\mathrm{d}z| \ll \frac{(e_rD(1))^{2\alpha}n}{K^2}.
\end{aligned}
$$

Collecting the last three estimates, by Lemma 2, we obtain the desired claim.

Lemma 5 is proved.

At this stage we have the following estimate.

Lemma 6. *If Condition (13) is satisfied and* $\delta n \ge 1$*, then there exists a positive constant* $c = c(d')$ *such that*

$$
F_m = J_0 + \mathrm{O}\big(e_r D_n \delta^c\big) \tag{18}
$$

uniformly in $0 \le r \le \delta n$ *and* $n/2 \le m \le n$*. Moreover,*

$$
D_n = \frac{1}{2\pi i n}\int_{\Delta_0} D'(z)\frac{\mathrm{d}z}{z^n} + \mathrm{O}\big(D_n\delta^c\big). \tag{19}
$$

Proof. It suffices to apply Lemmas 3 and 5 with $K = \delta^{c(\alpha)}n$, where

$$
c(\alpha) = \min\{1,\, d'(1-\alpha)/(\alpha d'+1)\},
$$

and optimize the function $d'\alpha c(\alpha)$ with respect to $\alpha \in (0,1)$. If $d' \le 3$, then (18) holds with $c = (\sqrt{1+d'}-1)^2$. If $d' > 3$, the choice $\alpha = (d'-1)/2d'$ gives $c(\alpha) = 1$; thus, (18) holds with $c = (d'-1)/2$. To obtain (19), use (18) with $r = 0$ and $m = n$.

The lemma is proved.

Lemma 7. *If* $0 \le \eta \le 1/2$ *and* $1/n \le \delta \le 1/2$ *are arbitrary, then*

$$
J_0 = e_r D_n\Big(1 + \mathrm{O}\Big(\big(\eta + (r/n)\mathbf{1}\{r \ge 1\}\big)\delta^{-1} + \delta^c\Big)\Big)
$$

uniformly in $n(1-\eta) \le m \le n$ *and* $0 \le r \le \delta n$ *with the constant* c *defined in Lemma 5.*

Proof. If $z \in \Delta_0$ and $r \ge 1$, then

$$
\begin{aligned}
F'(z) &= e_r D(z)\exp\Big\{-\sum_{j\le r}\frac{d_j}{j}(z^j-1)\Big\}q(z) \\
&= e_r D(z)\Big(1+\mathrm{O}\Big(\frac{r}{\delta n}\Big)\Big)\Big(\sum_{j\le n} - \sum_{j\le r}\Big)d_j z^{j-1} \\
&= e_r D'(z)\Big(1+\mathrm{O}\big(r/\delta n\big)\Big) + \mathrm{O}\big(r e_r D(1)\big)
\end{aligned}
$$

and

$$z^{-m} = z^{-n}\big(1 + \mathrm{O}(\eta\delta^{-1})\big).$$

Consequently, by virtue of $m^{-1} = n^{-1}\big(1+\mathrm{O}(\eta)\big)$, from Lemma 2 and Equation (19), we obtain

$$\begin{aligned} J_0 &= \frac{e_r}{2\pi i n}\Big(1+\mathrm{O}\Big(\Big(\frac{r}{n}+\eta\Big)\frac{1}{\delta}\Big)\Big)\int_{\Delta_0} D'(z)\frac{\mathrm{d}z}{z^n} + \mathrm{O}\Big(e_r D_n \frac{r}{\delta n}\Big) \\ &= e_r D_n\Big(1+\mathrm{O}\big((r/n+\eta)\delta^{-1}+\delta^c\big)\Big). \end{aligned}$$

If $r<1$, the terms having the fraction r/n do not appear.

The lemma is proved.

Proof of Proposition 1. Apply (18) and the last lemma.

Proof of Fundamental Lemma. We now apply Lemma 1 and Proposition 1 with $d_j = \lambda_j$. Condition (13) for weakly logarithmic assemblies is satisfied. From (12) and Proposition 1 with $\eta = (r/n)^{1/2}$ and $\delta = (r/n)^{1/2(1+c)}$, we obtain

$$\frac{P\big(\ell_{rn}(\bar\xi) = n-m\big)}{P\big(\ell(\bar\xi)=n\big)} = 1 + \mathrm{O}\big((r/n)^{c_0}\big), \quad c_0 := c/2(1+c),$$

uniformly in $0 \le m \le \sqrt{rn}$ provided that $1 \le r \le 4^{-1-c}n$.

The summands over $m > \sqrt{rn}$ in (11) contribute not more than

$$(rn)^{-1/2}\mathbf{E}\ell_r(\bar\xi) = (rn)^{-1/2}\sum_{j\le r} j\lambda_j \le \theta''(r/n)^{1/2}.$$

Hence, by (11), we obtain

$$\rho_{TV}\Big(\mathcal{L}\big(\bar k_r(\sigma)\big), \mathcal{L}(\bar\xi_r)\Big) \ll (r/n)^{c_1},$$

where $c_1 = \min\{1/2, c_0\}$ and $1 \le r \le 4^{-1-c}n$. Since the claim of Fundamental Lemma is trivial for $n \le 4^{1+c}$, we have finished its proof.

3 Proof of Theorem 2 and its Corollaries

Set $\mathbb{Z}_+^n(m) = \{\bar s \in \mathbb{Z}_+^n : \ \ell(\bar s) = m\}$ where $0 \le m \le n$. For arbitrary distributions $p_j(k)$, $1 \le j \le n$, on $\mathbb{Z}_+$ we define the product measure on $\mathbb{Z}_+^n$ by

$$P(\{\bar k\}) = \prod_{j\le n} p_j(k_j), \quad \bar k = (k_1, \dots, k_s) \in \mathbb{Z}_+^n.$$

Denote for brevity $P_n = P(\mathbb{Z}_+^n(n))$. Let $V = V(U)$ be the extension of an arbitrary subset $U \subset \mathbb{Z}_+^n$ defined in (4).

Lemma 8. *Suppose $n \ge 1$ and there exist positive constants c_2, c_3, C_1, C_2 such that*

(i) $p_j(0) \geq c_2$ *for all* $1 \leq j \leq n$;

(ii) $P\big(\mathbb{Z}_+^n(m)\big) \leq C_1 \left(\dfrac{n}{m+1}\right)^{1-\theta} P_n$ *for* $0 \leq m \leq n-1$ *and for some* $0 < \theta \leq 1$;

(iii) $P_n \geq c_3 n^{-1}$;

(iv) *for* $1 \leq m \leq n$,

$$\sum_{\substack{k \geq 1, j \leq n \\ kj = m}} \frac{p_j(k)}{p_j(0)} \leq \frac{C_2}{m}.$$

Then

$$P\big(\overline{V}\ |\mathbb{Z}_+^n(m)\big) \leq CP^\theta(\overline{U}) + C_1 C_2 \theta^{-1} n^{-\theta} \mathbf{1}\{\theta < 1\},$$

where

$$C := \max\left\{\frac{32}{c_2^2}, \frac{C_2}{c_3} + \frac{4C_1}{c_2} + \frac{C_1 C_2}{\theta}\right\}.$$

Proof. See [3], Appendix.

Proof of Theorem 1. It suffices to check conditions $(i)-(iv)$ of the last lemma for the poissonian probabilities $p_j(k)$ with parameters λ_j. By virtue of Condition (2), (i) and (iv) are trivial. Further, we find

$$\begin{aligned} P\big(\mathbb{Z}_+^n(m)\big) &= P\bigg(\sum_{j=1}^m j\xi_j = m, \xi_{m+1} = 0, \ldots, \xi_n = 0\bigg) \\ &= \exp\bigg\{-\sum_{j=1}^n \lambda_j\bigg\} \sum_{\ell_m(\bar{k})=m} \prod_{j=1}^m \frac{\lambda_j^{k_j}}{k_j!} \\ &= \exp\bigg\{-\sum_{j=1}^n \lambda_j\bigg\}[z^m] \exp\bigg\{\sum_{j\leq m} \lambda_j z^j\bigg\}, \quad 0 \leq m \leq n. \end{aligned}$$

Hence, applying Lemma 2, we obtain

$$P\big(\mathbb{Z}_+^n(m)\big) \asymp \frac{1}{m+1} \exp\bigg\{-\sum_{j=m+1}^n \lambda_j\bigg\}$$

for $0 \leq m \leq n$, where $a \asymp b$ means $a \ll b \ll a$. This and Condition (2) imply (ii) and (iii).

The theorem is proved.

Proof of Corollary 1. Apply Theorem 1 for

$$U = \Big\{\bar{t} \in \mathbb{Z}_+^n :\ H(\bar{t}) \in A\Big\},$$

where $H(\bar{t}) := \sum_{j\leq n} h_j(t_j)$, and check that

$$V(U) \subset \big\{\bar{s} \in \mathbb{Z}_+^n :\ H(\bar{s}) \in A + A - A\big\}.$$

Now

$$\begin{aligned}
\mu_n\big(h(\sigma) \notin A + A - A\big) &= P\big(H(\bar{\xi}) \notin A + A - A \,|\, \ell(\bar{\xi}) = n\big) \\
&\leq P\big(\bar{\xi} \notin V(U) \,|\, \ell(\bar{\xi}) = n\big) \\
&\ll P^\theta\big(\bar{\xi} \notin U\big) + \mathbf{1}\{\theta' < 1\} n^{-\theta'} \\
&= P^\theta\big(H(\bar{\xi}) \notin A\big) + \mathbf{1}\{\theta' < 1\} n^{-\theta'}.
\end{aligned}$$

Corollary 1 is proved.

Proof of Corollary 2. Apply the previous corollary for $\mathbb{G} = \mathbb{R}$ and $A = \{t : |t - a| \leq u/3\}$.

4 Proof of Theorem 2

We adopt the argument used in the case of permutations [15] and for the logarithmic assemblies [23].

Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables defined on some probability space $(\Omega, \mathcal{F}, P)$, with $\mathbb{E}Z_j = 0, \mathbf{E}Z_j^2 < \infty, j = 1, 2, \ldots,$ and

$$S_m = \sum_{j=1}^m Z_j, \qquad D_m^2 = \sum_{j=1}^m \mathbf{E}Z_j^2.$$

We define the polygonal lines $s_n(\cdot) : [0, D_n^2] \to \mathbb{R}$ such that

$$s_n(t) = S_m \frac{D_{m+1}^2 - t}{D_{m+1}^2 - D_m^2} + S_{m+1} \frac{t - D_m^2}{D_{m+1}^2 - D_m^2}$$

if $D_m^2 \leq t < D_{m+1}^2$ and $0 \leq m \leq n - 1$. Set also

$$S_n(t) = \frac{s_n(D_n^2 t)}{\sqrt{2D_n^2 LLD_n^2}}$$

for $0 \leq t \leq 1$ and $n \in \mathbb{N}$.

Lemma 9. *Let $D(n) \to \infty$ as $n \to \infty$. Assume that there exists a sequence*

$$M_n = \mathrm{o}\bigg(\frac{D_n}{\sqrt{LLD_n}}\bigg)$$

such that

$$P\big(|Z_n| \leq M_n\big) = 1$$

for each $n \geq 1$. Then

$$S_n(\cdot) \Rightarrow \mathcal{K} \quad (P\text{-}a.s.).$$

Proof. This is Major's Theorem [12].

We will apply Lemma 9 for $Z_j = a_j\big(\eta_j - (1-\mathrm{e}^{-\lambda_j})\big)$, where $\eta_j := \mathbf{1}\{\xi_j \ge 1\}$ and $1 \le j \le n$. Then $D_n^2 = B^2(n)$ and Condition (6) will be at our disposal. To simplify the calculations, we introduce another sequence of additive functions

$$\tilde h(\sigma, m) := \sum_{j=1}^{m} a_j \mathbf{1}\{k_j(\sigma) \ge 1\}, \quad m \le n.$$

Let $\tilde u_m(\sigma,t)$ and $\widetilde U_m(\sigma,t)$ be the combinatorial processes defined as $u_m(\sigma,t)$ and $U_m(\sigma,t)$ using $\tilde h(\sigma,m)$ instead of $h(\sigma,m)$. Set also $Y_m = a_1\eta_1 + \cdots + a_m\eta_m$ for $1 \le m \le n$.

Lemma 10. *For arbitrary* $\varepsilon > 0$*,*

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} \mu_n\Big(\max_{n_1 \le m \le n} \rho\big(\widetilde U_m(\sigma,\cdot), U_m(\sigma,\cdot)\big) \ge \varepsilon\Big) = 0. \tag{20}$$

Proof. If j and j' are the consecutive numbers from the set $I := \{j \le m : a_j \ne 0\}$, then, by virtue of the definition of $u_m(\sigma,t)$,

$$\begin{aligned} &\max\Big\{|\widetilde U_m(\sigma,t) - U_m(\sigma,t)| : \frac{B^2(j)}{B^2(m)} \le t \le \frac{B^2(j')}{B^2(m)}\Big\} \\ \le\ & \beta^{-1}(m)\max\Big\{|\tilde h(\sigma,j) - h(\sigma,j)|, |\tilde h(\sigma,j') - h(\sigma,j')|\Big\}. \end{aligned}$$

Hence

$$\begin{aligned} &\mu_n\Big(\max_{n_1\le m\le n} \rho\big(\widetilde U_m(\sigma,\cdot), U_m(\sigma,\cdot)\big) \ge \varepsilon\Big) \\ \le\ & \mu_n\Big(\max_{n_1\le m\le n}\max_{j\in I} |\tilde h(\sigma,j) - h(\sigma,j)| \ge \varepsilon\beta(n_1)\Big) \\ \le\ & \mu_n\Big(\sum_{j=1}^{n} \big|h_j(k_j(\sigma)) - a_j\cdot \mathbf{1}\{k_j(\sigma)\ge 1\}\big| \ge \varepsilon\beta(n_1)\Big) \\ \ll\ & P^\theta\Big(\sum_{j=1}^{n} \big|h_j(\xi_j) - a_j\eta_j \ge 1\}\big| \ge (\varepsilon/3)\beta(n_1)\Big) + \mathrm{o}(1). \end{aligned}$$

In the last step we applied Corollary 2. In its turn, if $K > 2$ is arbitrary, the probability appearing on the right-hand side can be majorized by

$$\begin{aligned} & P\big(\exists j \le K : \xi_j \ge K\big) + P\big(\exists j > K : \xi_j \ge 2\big) \\ +\ & P\Big(\sum_{j\le K}\big(|h_j(\xi_j)| + |a_j|\eta_j\big) \ge (\varepsilon/3)\beta(n_1), \quad 2 \le \xi_j \le K, \forall j \le K\Big). \end{aligned}$$

Since $\beta(n_1) \to \infty$ as $n_1 \to \infty$, the last probability is negligible. The first two of them do not exceed

$$\sum_{j\le K}\sum_{k\ge K}\frac{e^{-\lambda_j}\lambda_j^k}{k!} + \sum_{j\ge K}\sum_{k\ge 2}\frac{e^{-\lambda_j}\lambda_j^k}{k!} \ll K^{-1}.$$

Collecting the estimates, since K is arbitrary, we obtain the desired claim of Lemma 10.

In the sequel, we use only the functions $\tilde{h}(\sigma,m)$ and the processes $\widetilde{U}_m(\sigma,t)$ writing them without the "tilde".

Lemma 11. *Let $1\le k\le n$, $0<b_n\le b_{n-1}\le\cdots\le b_1$, and $\varepsilon>0$ be arbitrary. For $h=\tilde{h}$, if $n\to\infty$, we have*

$$\begin{aligned}
&\mu_n\Big(\max_{k\le m\le n} b_m\big|h(\sigma,m)-A(m)\big|\ge\varepsilon\Big)\\
\ll\ \ & P^\theta\Big(\max_{k\le m\le n} b_m\big|Y_m-A(m)\big|\ge\varepsilon/3\Big)+\mathrm{o}(1)\\
\le\ \ & 3^{2\theta}\varepsilon^{-2\theta}\bigg(b_k^2B^2(k)+\sum_{k\le j\le n} b_j^2a_j^2\mathrm{e}^{-\lambda_j}(1-\mathrm{e}^{-\lambda_j})\bigg)^\theta+\mathrm{o}(1).
\end{aligned}$$

Proof. The first estimate follows from Corollary 1 applied for $\mathbb{G}=\mathbb{R}^{n-r+1}$,

$$A=\Big\{(s_r,\dots,s_n)\in\mathbb{R}^{n-r+1}:\ \max_{r\le m\le n}|s_m-A(m)|<\varepsilon/3\Big\},$$

and

$$h(\sigma)=\big(h(\sigma,r),\dots,h(\sigma,n)\big).$$

The second inequality in Lemma 11 is just a partial case of Theorem 13 in Chapter III of [24].

The lemma is proved.

Let r, $n_1\le r\le n$, be a parameter, $q:=\max\{j\in I:\ j\le r\}$, and

$$u_m^{(r)}(\sigma,t)=\begin{cases} u_m(\sigma,t) & \text{if}\quad t\le B^2(q),\\ u_m(\sigma,B^2(q)) & \text{if}\quad t>B^2(q).\end{cases}$$

Denote $U_m^{(r)}(\sigma,t):=u_m^{(r)}\big(\sigma,B^2(m)t\big)/\beta(m)$. Similarly, let

$$s_m^{(r)}(t)=\begin{cases} s_m(t) & \text{if}\quad t\le B^2(q),\\ s_m\big(B^2(q)\big) & \text{if}\quad t>B^2(q)\end{cases}$$

and $S_m^{(r)}(t)=s_m^{(r)}\big(tB(m)\big)/\beta(m)$.

Lemma 12. *There exists a sequence $r=r(n)$, $n_1\le r=\mathrm{o}(n)$, such that, for every $\varepsilon>0$,*

$$\lim_{n_1\to\infty}\limsup_{n\to\infty} P\Big(\max_{n_1\le m\le n}\rho\big(S_m(\cdot),S_m^{(r)}(\cdot)\big)\ge\varepsilon\Big)=0 \tag{21}$$

and

$$\lim_{n_1\to\infty}\limsup_{n\to\infty}\mu_n\Big(\max_{n_1\le m\le n}\rho\big(U_m(\sigma,\cdot),U_m^{(r)}(\sigma,\cdot)\big)\ge\varepsilon\Big)=0.$$

Proof. If $P_{n_1,n}(\varepsilon)$ denotes the probability in (21) and $n_1 \le r \le n$, then

$$\begin{aligned}
&P_{n_1,n}(\varepsilon) = P\Big(\max_{r\le m\le n} \rho\big(S_m(\cdot), S_m^{(r)}(\cdot)\big) \ge \varepsilon\Big) \\
= \quad & P\Big(\max_{r\le m\le n} \frac{1}{\beta(m)} \sup\big\{\big|s_m(t) - s_m(B^2(q))\big| :\ B^2(q) \le t \le B^2(m)\big\} \ge \varepsilon\Big) \\
\le \quad & P\Big(\max_{r<m\le n} \beta^{-1}(m)\big|(Y_m - A(m)) - (Y_r - A(r))\big| \ge \varepsilon\Big) \\
\le \quad & \varepsilon^{-2}\,\frac{B^2(n) - B^2(r)}{\beta^2(r)}
\end{aligned}$$

by the already mentioned Theorem 13 [24], Chapter III.

The same argument and Lemma 11 (applied in the case $a_j \equiv 0$ if $j \le r$) leads to the estimate

$$\begin{aligned}
&\mu_n\Big(\max_{n_1\le m\le n} \rho\big(U_m(\sigma,\cdot), U_m^{(r)}(\sigma,\cdot)\big) \ge \varepsilon\Big) \\
\le \quad & \mu_n\Big(\max_{r<m\le n} \beta^{-1}(m)\big|(h(\sigma,m) - A(q)) - (h(\sigma,r) - A(r))\big| \ge \varepsilon\Big) \\
\ll \quad & P^{\theta}\Big(\max_{r<m\le n} \beta^{-1}(m)\big|(Y_m - A(m)) - (Y_r - A(r))\big| \ge (1/3)\varepsilon\Big) + \mathrm{o}(1) \\
\ll \quad & \left(\frac{B^2(n) - B^2(r)}{\beta^2(r)}\right)^{\theta} + \mathrm{o}(1)
\end{aligned}$$

as $n \to \infty$.

By Condition (4), if r is sufficiently large, $r \le j \le n$, and δ, $0 < \delta < 1$, is arbitrary, then $|a_j| \le \delta B(n)/\sqrt{LLB(n)}$. Hence, taking $r = \delta n$ and applying Condition (13), we obtain

$$B^2(n) - B^2(r) \ll \delta^2 \log\frac{1}{\delta}\,\frac{B^2(n)}{LLB(n)}.$$

We now choose $\delta = \delta_n = \mathrm{o}(1)$ as $n \to \infty$ so that $\delta \ge 1/\sqrt{n}$. This implies $B^2(n) - B^2(r) = \mathrm{o}(\beta^2(r))$. Having in mind the above estimates, we see that, with such an r, the probabilities in Lemma 12 vanish as $n \to \infty$ and $n_1 \to \infty$.

The lemma is proved.

Proof of Theorem 2. By virtue of the definition of strong convergence and Lemma 12, it suffices to prove that

$$\lim_{n_1\to\infty} \limsup_{n\to\infty} \mu_n\Big(\max_{n_1\le m\le n} \rho\big(U_m^r(\sigma,\cdot), \mathcal{K}\big) \ge \varepsilon\Big) = 0$$

and

$$\lim_{n_1\to\infty} \liminf_{n\to\infty} \mu_n\Big(\min_{n_1\le m\le n} \rho\big(U_m^r(\sigma,\cdot), g\big) < \varepsilon\Big) = 1$$

for each function $g \in \mathcal{K}$ and $\varepsilon > 0$. Since here $r = r(n) \to \infty$ and $r = \mathrm{o}(n)$, we can apply the Fundamental Lemma and substitute the frequencies by the

appropriate probabilities for independent r.vs. Consequently, our task reduces to the proof of

$$\lim_{n_1\to\infty}\limsup_{n\to\infty} P\Big(\max_{n_1\le m\le n}\rho\big(S_m^r(\cdot),\mathcal{K}\big)\ge\varepsilon\Big)=0$$

and

$$\lim_{n_1\to\infty}\liminf_{n\to\infty} P\Big(\min_{n_1\le m\le n}\rho\big(S_m^r(\cdot),g\big)<\varepsilon\Big)=1.$$

Checking that the last relations follow from Lemmas 9 and 12 we complete the proof of Theorem 2.

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